

Lie Groups and Lie Algebras, MMath-Midterm, Sep-2026

All intermediate steps in your proofs should be explained. If you use a theorem proved in class, you must state it before applying it, unless the question itself asks you to prove that theorem.

All 7 questions carry 5 marks each, you may attempt all. Max marks: 30

1. Let $f: G \rightarrow H$ be a Lie-group homomorphism between connected Lie groups. Suppose that

$$df_e: \mathfrak{g} \longrightarrow \mathfrak{h}$$

is an isomorphism. Show that f is a covering map and that $\ker f$ is a discrete subgroup contained in the centre $Z(G)$.

2. Prove the exceptional isomorphism

$$\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}(2, 1)^\circ,$$

where $\mathrm{SO}(2, 1)^\circ$ denotes the identity component of $\mathrm{SO}(2, 1)$.

3. Let $n \geq 2$. Show that $\mathrm{SU}(n)$, with its usual real Lie-group structure, cannot be given the structure of a complex Lie group.
4. Show that every finite-dimensional complex representation of the real Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is a direct sum of irreducible representations.
5. For each $n \geq 1$, let

$$\mathfrak{s}(n, \mathbb{R}) = \left\{ X \in \mathfrak{gl}_n(\mathbb{R}) : X = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\},$$

the Lie algebra of upper-triangular $n \times n$ matrices. Consider the vector space

$$\mathfrak{g} = \bigoplus_{n=1}^{\infty} \mathfrak{s}(n, \mathbb{R}),$$

whose elements have only finitely many nonzero coordinates.

Define a Lie bracket on $\mathfrak{g}$ such that every $\mathfrak{s}(n, \mathbb{R})$ is an ideal. Show that every element of $\mathfrak{g}$ is contained in a finite-dimensional solvable ideal, but that the derived series of $\mathfrak{g}$ does not terminate at 0. Deduce that $\mathfrak{g}$ has no largest solvable ideal.

6. (a) Let $X \in \mathfrak{sl}_2(\mathbb{R})$ have a nonreal eigenvalue. Show that its eigenvalues are ia and $-ia$, for some $a \in \mathbb{R} \setminus \{0\}$, and that there exists $C \in \mathrm{GL}_2(\mathbb{R})$ such that

$$X = C \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} C^{-1}.$$

- (b) Let $A \in \mathrm{SL}_2(\mathbb{R})$ have a nonreal eigenvalue. Show that there exist $C \in \mathrm{GL}_2(\mathbb{R})$ and $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$ such that

$$A = C \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} C^{-1}.$$

(c) By also considering the cases of real and repeated eigenvalues, show that

$$\exp(\mathfrak{sl}_2(\mathbb{R})) = \{A \in \mathrm{SL}_2(\mathbb{R}) : \mathrm{tr}(A) > -2\} \cup \{-I\}.$$

7. (a) Show that the indefinite orthogonal group $\mathrm{O}(1, 1)$ has exactly four connected components.

(b) Show that the exponential map

$$\exp: \mathfrak{u}(n) \longrightarrow U(n)$$

is surjective but not one-to-one.